



RESEARCH REPORT

On Weakly Mixing Transformation Semigroups

P. F. Renaud

No. 21

February 1982

Department of Mathematics

**University of Canterbury
Christchurch
New Zealand**

ON WEAKLY MIXING TRANSFORMATION SEMIGROUPS

by

P.F. Renaud

ABSTRACT

A generalization of the classical ergodic mixing theorem is given, valid for arbitrary topological semigroups. Conditions under which the tensor product of two representations has no fixed points are obtained and these yield simple necessary and sufficient conditions for the Cartesian product of two measure preserving transformations to be ergodic.

1. INTRODUCTION

Let (X, S, μ) be a finite measure space with $\mu(X) = 1$ and T be a measure-preserving transformation on X . T is called weakly mixing if

$$(i) \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} |\mu(E \cap T^{-j}F) - \mu(E)\mu(F)| = 0$$

for all measurable sets E, F . If T is invertible then the ergodic mixing theorem (see Halmos [6, p.39]) establishes the equivalence of (i) with each of the following two conditions:

(ii) The unitary operator U on $L_2(X)$ induced by T has continuous spectrum on the subspace of $L_2(X)$ orthogonal to the constant functions.

(iii) The cartesian product $T \times T$ on the product space $(X \times X, S \times S, \mu \times \mu)$ is ergodic.

In [3], H.A. Dye obtained an abstract mixing theorem considerably more general than the above. Instead of the semi-group of non-negative integers he considered an arbitrary amenable topological semigroup. The operation $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1}$ was replaced by almost convergence and the condition that U has continuous spectrum was replaced by the condition that the representation U_g should have no finite-dimensional subrepresentation.

More recently, B. Kronfeld [7] has considered the Cartesian product $S \times T$ of two invertible measure-preserving transformations and has shown that if S and T have discrete spectrum then a necessary and sufficient condition that $S \times T$ be ergodic is that the spectra have trivial intersection. The aim of this work is two-fold. Firstly we show that the amenability condition assumed by Dye may be dispensed with, and obtain a mixing theorem valid for arbitrary topological semigroups. Secondly by considering the

tensor product of two possibly different representations we generalize and somewhat strengthen Kronfeld's result. To avoid repetitious arguments we obtain both results as one theorem. This has one minor disadvantage. To obtain the required generalization of Kronfeld's theorem we apparently have to assume some invertibility conditions on the representations. These are not necessary in the case of one repeated representation but nonetheless will be assumed.

A brief note on amenability seems to be called for. The equivalence between amenability and the fixed-point property (see e.g. Greenleaf [5], p. 49) shows that the validity of various ergodic theorems on general Banach spaces depends strongly on the amenability condition. However if we restrict our representations to Hilbert space (or more generally to uniformly convex spaces) the amenability assumption may then be safely dropped. This distinction is more explicitly made in [1].

2. PRELIMINARY RESULTS

Dye obtained his mixing theorem by extending the traditional Peter-Weyl theory. Our arguments are more elementary, relying basically on an ergodic theorem of Birkhoff and properties of positive-definite functions. Since however we deal with semigroups rather than groups, some minor technical difficulties (not present in the group case) arise.

By a topological semigroup, we mean a semigroup S which is also a topological space such that multiplication is separately continuous.

The following ergodic theorem is due to Birkhoff [2] and is

analogous to Day's fixed point theorem without the amenability condition.

Theorem 2.1. Let H be a Hilbert space, S a topological semigroup and $g \rightarrow U_g$ a weakly continuous representation or anti-representation of S on H such that $\|U_g\| \leq 1$ for all $g \in S$. Let $M = \{x \in H : U_g x = x \text{ for all } g \in S\}$ and for $x \in H$, let K_x be the closed convex hull of $\{U_g x : g \in S\}$. Then for all $x \in H$, $K_x \cap M$ consists of exactly one point which is simultaneously

- (i) the projection of x on M
- (ii) the point of K_x with minimum norm.

Positive-definite functions on semigroups may be introduced in the following way.

Definition 2.2. Let S be a topological semigroup. A complex-valued function ϕ on S is called positive-definite if there exists a Hilbert space H , a (weakly) continuous isometric representation $g \rightarrow U_g$ of S on H and a vector $x \in H$ such that $\phi(g) = (U_g x, x)$ for all $g \in S$.

We write $\phi \leftrightarrow (H, U_g, x)$ and note that $\phi \in CB(S)$, the Banach algebra of continuous, bounded functions on S . Denote by P the set of positive definite functions on S . For groups, this definition is equivalent to the usual one. In particular if x is cyclic then the representation $g \rightarrow U_g$ is uniquely determined to within unitary equivalence. For semigroups this does not seem to be the case since we only assume U_g to be isometric. So any properties of ϕ which depend on the particular representation will have to be shown properly defined.

The following lemmas give the properties of positive-definite

functions which we will require. They generalize to semigroups the well-known properties of these functions on groups. For proofs in the case where S is a group, we refer the reader to [4].

Lemma 2.3. *Let $\phi, \psi \in P$, $\alpha \geq 0$. Then $\alpha\phi$, $\phi + \psi$, $\phi\psi$ and ϕ all $\in P$. Further, P contains all the non-negative constants.*

The proofs of the first two properties imply that every finite linear combination of positive-definite functions is of the form $(U_g x, y)$. This in turn implies

Lemma 2.4. *The set V of finite linear combinations of positive-definite functions is a left and right translation invariant subalgebra of $CB(S)$, closed under complex conjugation and containing the constants.*

Now let $f \in CB(S)$. Denote by K_f^1 the (norm)-closed convex hull of $\{ {}_h f : h \in S \}$ and by K_f^r the (norm)-closed convex hull of $\{ f_h : h \in S \}$. (Here ${}_h f$, f_h are defined by ${}_h f(g) = f(hg)$ and $f_h(g) = f(gh)$).

Definition 2.5. f is called left (right) ergodic if K_f^1 (K_f^r) contains a constant function. f is called ergodic if it is both left and right ergodic.

Lemma 2.6. *If f is ergodic then K_f^1 and K_f^r contain precisely one and the same constant denoted by $m(f)$.*

Lemma 2.7. *Every positive-definite function is ergodic. If $\phi \in P$ with $\phi \leftrightarrow (H, U_g, x)$ then $m(\phi) = \|Px\|^2$ where P is the projection onto the subspace M of invariant elements of H .*

This lemma can now be applied to prove the existence of an invariant mean on V .

Lemma 2.8. *There exists a unique linear functional m on V satisfying*

- (i) $|m(\phi)| \leq \|\phi\|_\infty$
- (ii) $m(1) = 1$
- (iii) $m({}_g\phi) = m(\phi_g) = m(\phi)$ for all $g \in S$
- (iv) $m(\phi) \geq 0$ if $\phi \in P$ or if $\phi \geq 0$.

Remark. The properties (i)-(iv) are proved for groups in [4].

To prove uniqueness we argue as follows: let m' be any linear functional on V satisfying (i)-(iv). Define

$\langle x, y \rangle = m'[(U_g x, y)]$ where $g \rightarrow U_g$ is some isometric representation of S on H . Then \langle, \rangle is a bounded bilinear form on H so that there exists a bounded linear operator A on H with $\langle x, y \rangle = (Ax, y)$. A is self-adjoint by (iv) and by (iii) we have $AU_g = A = U_g A$ for all $g \in S$.

For $x \in H$, $\alpha_1, \dots, \alpha_n > 0$, $\sum \alpha_i = 1$, $g_1, \dots, g_n \in S$ we have

$$A(\sum \alpha_i U_{g_i} x) = Ax$$

so that by continuity $Ay = Ax$ for all $y \in K_x$. In particular with $y = Px$ (P the projection onto the invariant subspace) we obtain $AP = A = PA$. But since $AU_g = A = U_g A$ it follows from a standard separation theorem that for all x , $Ax \in K_x$. Hence we may use the fact that $P(\sum \alpha_i U_{g_i} x) = Px$ to obtain $PA = P = AP$. Hence $A = P$ and $m' = m$. So m is unique.

Now let $g \rightarrow U_g$, $g \rightarrow V_g$ be two weakly continuous isometric representations of S on Hilbert spaces H , H' respectively.

Fix $x_0 \in H$, $y'_0 \in H'$ and for $x \in H$, $y \in H'$ define

$$\langle x, y' \rangle = m[(x, U_h x_0) (V_h y'_0, y')]$$

[Note that by lemma 2.3, $\phi(h) = (x, U_h x_0) (V_h y'_0, y') \in V$ so that $\langle x, y' \rangle$ is well-defined]. Then \langle, \rangle is a bounded bilinear form on $H \times H'$ so that there exists a bounded linear operator $A : H \rightarrow H'$ such that

$$(Ax, y') = m[(x, U_h x_0) (V_h y'_0, y')]$$

Proposition 2.9. *A is a compact operator which satisfies*

$$AU_g = V_g A \text{ for all } g \in S.$$

$$\begin{aligned} \text{Proof. For } g \in S, (AU_g x, y') &= m_h[(U_g x, U_h x_0) (V_h y'_0, y')] \\ &= m_h[(U_g x, U_{gh} x_0) (V_{gh} y'_0, y')] \end{aligned}$$

(by translation invariance of m).

$$\begin{aligned} &= m_h[(x, U_h x_0) (V_h y'_0, V_g^* y')] \\ &= (Ax, V_g^* y') \\ &= (V_g Ax, y') \end{aligned}$$

$$\text{So } AU_g = V_g A.$$

To show A compact, let $\{x_n\} \subset H$ be weakly convergent to 0.

It suffices to show that $\lim ||Ax_n|| = 0$. We have

$$\begin{aligned} ||Ax_n||^2 &= (Ax_n, Ax_n) \\ &= m[(x_n, U_h x_0) (V_h y'_0, Ax_n)] \\ &= m(\phi_n(h)) \text{ say.} \end{aligned}$$

Now let J be a conjugation on H , i.e. J is a conjugate linear

operator such that $J^2 = I$ and $(Jx, Jy) = (y, x)$ for $x, y \in H$.

For $g \in S$ define $U_g^J = JU_gJ$. Then $g \rightarrow U_g^J$ is a weakly continuous isometric representation of S and

$$\begin{aligned}\phi_n(h) &= (x_n, U_h x_0) (V_h y_0', Ax_n) \\ &= (U_h^J Jx_0, Jx_n) (V_h y_0', Ax_n) \\ &= ((U_h^J \otimes V_h) (Jx_0 \otimes y_0'), Jx_n \otimes Ax_n)\end{aligned}$$

so that using lemma 2.7,

$$||Ax_n||^2 = (Q Jx_0 \otimes y_0', Jx_n \otimes Ax_n)$$

where Q is the projection onto the subspace of $H \otimes H'$ of all elements invariant under the representation $g \rightarrow U_g^J \otimes V_g$. But since $\{x_n\}$ is weakly convergent to 0 in H , it follows that $\{Jx_n \otimes Ax_n\}$ is weakly convergent to 0 in $H \otimes H'$. i.e. $\lim ||Ax_n||^2 = 0$ and A is compact.

3. The mixing theorem.

Theorem 3.1. Let $g \rightarrow U_g$, $g \rightarrow V_g$ be weakly continuous unitary representations of S on H , H' respectively and let J be a conjugation of H . Let Q be the projection onto the subspace of $H \otimes H'$ of all elements invariant under $g \rightarrow U_g^J \otimes V_g$. Then the following conditions are equivalent:

- (i) $m[(U_g x, y) (V_g x', y')] = 0$ for all $x, y \in H$; $x', y' \in H'$.
- (ii) U_g and V_g have no non-zero unitarily equivalent finite-dimensional subrepresentations.
- (iii) $Q = 0$.

Proof. (i) \rightarrow (iii). Assuming (i) we have for all $x, y \in H$, $x', y' \in H'$,

$$\begin{aligned}
0 &= m[(U_g^J Jx, Jy)(V_g x', y')] \\
&= m[((U_g^J \otimes V_g)(Jx \otimes x'), Jy \otimes y')]
\end{aligned}$$

from which it follows by linearity and continuity that for all $u, v \in H \otimes H'$,

$$\begin{aligned}
m[(U_g^J \otimes V_g)u, v] &= 0 = (Qu, v) \\
\text{i.e. } Q &= 0
\end{aligned}$$

(iii) \rightarrow (i) is obtained by an obvious reverse argument.

(i) \rightarrow (ii). If (ii) fails then there exist non-zero finite-dimensional subspaces M, M' of H, H' respectively, and a unitary operator $W : M \rightarrow M'$ such that for all $g \in S$, $U_g : M \rightarrow M'$ and $WU_g = V_g W$, let $x, y \in M$ and put $x' = Wx, y' = Wy$. By (i)

$$\begin{aligned}
0 &= m[(\overline{U_g x, y})(V_g x', y')] \\
&= m[|(U_g x, y)|^2]
\end{aligned}$$

But since M is finite-dimensional, the function $(U_g x, y)$ and hence $|(U_g x, y)|^2$ is almost periodic. Hence $(U_g x, y) = 0$ for all x, y i.e. $M = (0)$ a contradiction.

(ii) \rightarrow (i). Fix $x_0 \in H, y_0 \in H'$ and as before define $A : H \rightarrow H'$ by

$$(Ax, y') = m[(x, U_h x_0)(V_h y_0', y')]$$

By proposition 2.9, A is compact and

$$AU_g = V_g A \quad \text{for } g \in S.$$

By hypothesis U_g and V_g are unitary operators so that moreover

$$U_g A^* = A^* V_g$$

from which we deduce that

$$A^*AU_g = U_gA^*A \quad \text{and} \quad AA^*V_g = V_gA^*A$$

We can write $A = UH$ where H is the positive square root of A^*A and U a partial isometry with associated subspaces $X = \text{cl}(\text{range}(A^*))$ and $X' = \text{cl}(\text{range}(A))$. Then

$$AA^* = U(A^*A)U^*$$

and since A is compact, so are AA^* and A^*A .

Writing $A^*A = \sum \lambda_i P_i$ for the spectral decomposition of A^*A we have $AA^* = \sum \lambda_i Q_i$ for the decomposition of AA^* where $Q_i = U P_i U$ and the projections P_i and Q_i are finite dimensional if $\lambda_i \neq 0$. Moreover for all g , $U_g P_i = P_i U_g$ and $V_g Q_i = Q_i V_g$. But this means that U_g and V_g have unitarily equivalent (via U) finite dimensional subrepresentations. By hypothesis then, $A^*A = 0$ i.e. $A = 0$ and (i) is proved.

Remark. If $U_g = V_g$ then it suffices to assume that U_g is an isometric representation since the operator A may be chosen self-adjoint. In this case condition (i) may be replaced by

$$(i') \quad m[|(U_g x, y)|^2] = 0 \quad \text{for all } x, y \in H.$$

In Dye's original theorem ([3], theorem 1) where S is assumed amenable (i') reads

$$(i'') \quad |(U_g x, y)| \text{ is almost convergent to } 0 \text{ for all } x, y \in H.$$

The equivalence between (i') and (i'') follows from the uniqueness of the invariant mean m on the space V together with the fact that if n is a mean on $CB(S)$ then

$$n(|f|^2) = 0 \Leftrightarrow n(|f|) = 0$$

(since it is easily shown that $[n(|f|)]^2 \leq n(|f|^2) \leq |f| n(|f|)$).

We now indicate how to obtain a more concrete form of the mixing theorem. Suppose that (X, S, μ) is a finite measure space with $\mu(X) = 1$ and that S is realized as a semigroup of invertible measure-preserving transformations on X under the map $x \rightarrow xS_g$ in such a way that $g \rightarrow \mu(E \cap FS_g^{-1})$ is continuous for all $E, F \in S$. Let U_g denote the associated unitary operator on $L_2(X)$. Then $g \rightarrow U_g$ is a weakly continuous unitary anti-representation of S on $L_2(X)$. Similarly let $x \rightarrow xT_g$ with associated V_g define another anti-representation. The following conditions are then equivalent

- (i) $m[\mu(A \cap BS_g^{-1})\mu(C \cap DT_g^{-1})] = 0$ for $A, B, C, D \in S$
- (ii) the only non-zero unitarily equivalent finite-dimensional subrepresentations of U_g and V_g are their restrictions (the identity) to the subspace of constant functions.
- (iii) the semigroup of product transformations $S_g \times T_g$ on the product space $X \times X$ is ergodic.

We omit the proof noting merely that it follows from theorem 3.1 precisely as corollary 1 follows from theorem 1 in [3] and that theorem 3.1 applies also to anti-representations. Specializing to the case of two invertible measure-preserving transformations, conditions (ii) and (iii) now imply the following corollary which generalizes Kronfeld's results in [7] and answers a conjecture he made.

Corollary 3.2. Let S, T be ergodic invertible measure-preserving transformations on a finite measure space and let $p\sigma(S), p\sigma(T)$ denote their point spectra. Then a necessary and sufficient condition that $S \times T$ is ergodic is that $p\sigma(S) \cap p\sigma(T) = \{1\}$. In particular if S is weakly mixing then $S \times T$ is ergodic for all ergodic T .

In this last case even more can be said. I wish to thank Dr. P.H. Butler for pointing out the plausibility of the following

Theorem 3.3. Let U_g, V_g be unitary representations of S on H, H' . If U_g has no finite-dimensional subrepresentations, then neither does $U_g^J \otimes V_g$.

Proof. By hypothesis $m[|(U_g x, y)|^2] = 0$ for all $x, y \in H$. It suffices to show that $m[|((U_g^J \otimes V_g)u, v)|^2] = 0$ for all $u, v \in H \otimes H'$ and by linearity and continuity we need only show that

$$m[|((U_g^J \otimes V_g)(Jx \otimes x'), Jy \otimes y'))|^2] = 0 \text{ for } x, y \in H, x', y' \in H'.$$

$$\text{i.e. } m[|(\overline{U_g x, y})(V_g x', y')|^2] = 0$$

But this follows easily from lemma 2.8

Corollary 3.4. If S and T are invertible measure-preserving and if S is weakly mixing, T ergodic then $S \times T$ is weakly mixing.

REFERENCES

1. L. Alaoglu and G. Birkhoff, *General ergodic theorems*, Ann. of Math. 41 No. 2 (1940), 293-309.
2. G. Birkhoff, *An ergodic theorem for general semigroups*, Proc. Nat. Acad. Sci. U.S.A. 25(1939), 625-627.
3. H.A. Dye, *On the ergodic mixing theorem*, Trans. Amer. Math. Soc. 118(1965), 123-130.
4. R. Godement, *les fonctions de type positif et la théorie des groupes*, Trans. Amer. Math. Soc. 63(1948), 1-84.

5. F.P. Greenleaf, *Invariant means on topological groups and their applications*, Van Nostrand, New York, 1969.
6. P. Halmos, *Ergodic theory*, Publ. Math. Soc. Japan, No. 3, Math. Soc. Japan, Tokyo, 1956.
7. B. Kronfeld, *Ergodic transformations and independence*, Glasnik Mat. 5(25) No. 2 (1970), 221-226.

University of Canterbury,
Christchurch, New Zealand.